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The bivector Clifford algebra and the geometry of Hodge dual operators

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Abstract. This article describes features of a natural representation of the Clifford algebra of the space of bivectors of a four-dimensional vector space, the representation space being the vector space plus its dual. Vectors and covectors are then pure spinors. The natural map taking a pure spinor to a totally null 3-bivector is shown to be intertwined with the operation of lowering or raising by a metric with the action of the Hodge dual operator. New formulae exhibiting a metric in terms of its dual operator are presented, as are a few applications.

1. Introduction

Given a four-dimensional (real) vector space with volume form, we may consider the space of bivectors with its (split signature) metric determined by the volume form. The Clifford algebra of this bivector space admits a natural irreducible representation on the eight-dimensional direct sum of the vector space and its dual. Urbantke has pointed out in a private communication (relayed by Jacobson) that his formula, (Urbantke 1984, Capovilla *et al* 1991a) expressing a metric in terms of a basis for the self-dual subspace of the Hodge dual operator, may be usefully derived from the point of view of this representation. The metric appears in the representative of the unit three-volume element of the self-dual subspace. We give an account of this and in addition develop the spinorial geometry of the representation in order to see precisely the geometric relation between a metric and its dual operator. The pure spinors of the representation are just the vectors and the covectors. They determine totally null 3-planes in the bivector space; in fact, they determine representative 3-bivectors. Given a metric on the original vector space, its lowering and raising action on pure spinors is seen to correspond to the action of the dual operator on totally null 3-bivectors. This correspondence leads to formulae giving a metric in terms of its dual operator.

The relation between a metric and its dual operator is of some significance in recent formulations of general relativity without the metric (Capovilla *et al* 1991b). The bivector Clifford algebra also provides a larger context for the (self-dual) 'bivector formalism' for four-dimensional Lorentzian geometry, which has been used in connection with classification of Weyl curvature tensors and the study of null congruences (e.g. Israel 1979). Because of the essential equivalence of metrics with dual operators (in the presence of a volume form), one could base four-dimensional geometry on

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the bivector bundle with a dual operator rather than on the tangent bundle with a metric. In the case that the original vector space is complex, some of the geometric objects naturally associated with the bivector Clifford algebra (e.g. totally null 'α-planes') are commonplace in twistor theory (Penrose and Rindler 1986). However, the algebra itself and its representation are usually not made explicit in this context. Here we observe that a non-degenerate symmetric 2-index twistor is in fact a 'metric' and the self-dual subspace of its associated dual operator determines the world line of the particle represented by the twistor. In any case, the bivector Clifford algebra, together with its natural representation, is an elegant and effective algebra associated with a four-dimensional vector space.

2. The vector–covector representation

We begin by exhibiting the representation of the Clifford algebra on the vector space plus its dual, together with some standard attendant constructions. These include the spinor inner product and the isometry of the exterior algebra with the spinor endomorphisms. For a discussion of these constructions in a general setting one may consult Harvey (1990)—to whom we shall appeal frequently—or the appendix of Penrose and Rindler (1986). A result specific to the present setting is the isomorphism of the spin group of the bivectors with the special linear group of the vectors.

Let V be a four-dimensional real vector space with a volume form $\varepsilon \in \Lambda^4 V^*$. (For now we only consider the real case; there is little to change for the complex case.) The space of bivectors $\mathcal{W} := \Lambda^2 V$ carries an inner product h of signature type (3,3) given by

$$h(F,G) := \tfrac{1}{2}\varepsilon(F \wedge G) \qquad \forall\ F,G \in \mathcal{W}\,.$$

A bivector F is null with respect to h iff $F \wedge F = 0$ iff F is simple ($F = u \wedge v$ for some $u, v \in V$). The Clifford algebra $\mathcal{C}$ of $(\mathcal{W}, h)$ has, up to equivalence, a unique faithful irreducible representation (of dimension eight). It is a pleasant fact that $V \oplus V^*$ is a natural representation space for $\mathcal{C}$.

To describe and use this representation it will be convenient to have recourse to abstract indices (Penrose and Rindler 1984); for V we employ the labelling set $\{a, b, c, d\}$ and for $\mathcal{W}$ we employ $\{i, j, k, i_1\}$. However, in the beginning, we accommodate readers tuned to index-free descriptions, following the tensor algebra conventions of Sternberg (1983), which accord with those of Penrose and Rindler. In particular, $u \wedge v = \frac{1}{2}(u \otimes v - v \otimes u)$.

Consider $\mathcal{W} = \Lambda^2 V$ as a subspace of $V \otimes V$ and identify $\Lambda^2 V^*$ with $\mathcal{W}^*$ via the pairing $\Lambda^2 V^* \times \Lambda^2 V \rightarrow \mathbf{R}$:

$$\langle \phi \wedge \psi \mid u \wedge v \rangle = \phi(u)\psi(v) - \phi(v)\psi(u)\,.$$

In index notation, $\langle \Phi \mid F \rangle = 2\Phi_{ab} F^{ab}$. Write Γ for the natural map $\mathcal{W} \rightarrow \mathrm{Hom}(V^*, V)$ given by

$$\langle \phi,\, \Gamma(F)\psi \rangle = \langle \phi \wedge \psi \mid F \rangle \qquad (\phi, \psi \in V^*)\,.$$

In index notation, $\Gamma_i{}^{ab} F^i \psi_b := 2F^{ab}\psi_b$. For $F \in \mathcal{W}$ define $\check{F}$ by

$$\langle \check{F} \mid H \rangle = 2h(F,H) = \varepsilon(F \wedge H)$$

and write $\check{\Gamma}$ for the map $\mathcal{W} \to \mathrm{Hom}(V, V^*)$ given by

$$\langle \check{\Gamma}(F)v,\, u\rangle = \langle \check{F} \mid u \wedge v\rangle \qquad (u, v \in V)\,.$$

In index notation, $\check{F}_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}$ and $\check{\Gamma}_{iab}F^i := 2\check{F}_{ab}$. Then the identity

$$\Gamma(F)\check{\Gamma}(F) = -h(F,F)\mathbf{1}_V \qquad (F \in \mathcal{W})$$

or $\Gamma_{(i}{}^{ab}\check{\Gamma}_{j)bc} = -h_{ij}\delta^a_c$, holds. (The index-free proof uses $2F \wedge (i_\phi F) \wedge v = -(F \wedge F)i_\phi v$, where $i_\phi F$ is the interior product.)

It follows that the map

$$\gamma : \mathcal{W} \to \mathrm{End}\,\mathcal{S} \qquad \mathcal{S} := V \oplus V^* \tag{2.1}$$

defined by

$$\gamma(F)\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} \Gamma(F)\phi \\ \check{\Gamma}(F)v \end{pmatrix}$$

is a Clifford map for h:

$$\gamma(F)^2 = -h(F,F)\,\mathbf{1}_{\mathcal{S}}\,.$$

The unique extension of (2.1) to the Clifford algebra $\mathcal{C}$ as an algebra homomorphism yields a faithful irreducible representation

$$\gamma : \mathcal{C} \to \mathrm{End}\,\mathcal{S}\,. \tag{2.2}$$

We shall take the set $\mathcal{C}$ to be the exterior algebra $\bigwedge \mathcal{W}$, to which $\mathcal{C}$ is naturally isomorphic as a vector space. The restriction of (2.2) to $\mathcal{W}$ is written

$$\gamma_i = \begin{pmatrix} 0 & \Gamma_i{}^{ab} \\ \check{\Gamma}_{iab} & 0 \end{pmatrix}\,.$$

The restrictions of (2.2) to $\bigwedge^2 \mathcal{W}$ and $\bigwedge^3 \mathcal{W}$, respectively, determine 'Cartan forms'

$$\gamma_{ij} := \gamma_{[i}\gamma_{j]} \equiv \begin{pmatrix} \Gamma_{ij}{}^a{}_b & 0 \\ 0 & \check{\Gamma}_{ija}{}^b \end{pmatrix} \tag{2.3}$$

and

$$\gamma_{ijk} := \gamma_{[i}\gamma_j\gamma_{k]} \equiv \begin{pmatrix} 0 & \Gamma_{ijk}{}^{ab} \\ \check{\Gamma}_{ijkab} & 0 \end{pmatrix} \tag{2.4}$$

this last matrix having entries symmetric in ab. According to (2.3), the even subalgebra of $\mathcal{C}$ lies in the direct sum $\mathrm{End}\,V \oplus \mathrm{End}\,V^*$; thus V and V^* are 'reduced' spinor spaces for the bivectors.

The 'hat involution' of $\mathcal{C}$

$$X \equiv X_1 \ldots X_p \mapsto \hat{X} = (-1)^p X_p \ldots X_1 = (-1)^{p(p+1)/2} X \tag{2.5}$$

(where $X_i \in \mathcal{W}$), gives us a hat involution on $\mathrm{End}\,\mathcal{S}$ via the isomorphism γ; thus $\widehat{\gamma(X)} = \gamma(\hat{X})$. Up to scale, there is a unique inner product $\mathbf{E}$ on $\mathcal{S}$ such that

$$\mathbf{E}(\mathbf{A}\xi, \zeta) = \mathbf{E}(\xi, \hat{\mathbf{A}}\zeta) \qquad \forall \mathbf{A} \in \mathrm{End}\,\mathcal{S}\,.$$

A natural choice of scale gives

$$\mathbf{E}(u+\phi, u+\phi) = 2\phi(u) \qquad \forall u \in V,\ \phi \in V^*\,.$$

As a map from $\mathcal{S}$ to $\mathcal{S}^*$, $\mathbf{E}$ is just $(u,\phi) \mapsto (\phi, u)$. The adjoint of $\mathbf{A} \in \mathrm{End}\,\mathcal{S}$ with respect to $\mathbf{E}$ is then

$$\hat{\mathbf{A}} = \mathbf{E}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{E} \tag{2.6}$$

where $\mathbf{A}^{\mathrm{T}} \in \mathrm{End}\,\mathcal{S}^*$ is the dual map.

The inner product h on $\mathcal{W}$ extends to an inner product on $\bigwedge \mathcal{W}$ given by

$$h(X|Y) := p!\, h_{i_1 j_1} \ldots h_{i_p j_p} X^{i_1 \ldots i_p} Y^{j_1 \ldots j_p} \qquad \forall X, Y \in \textstyle\bigwedge^p \mathcal{W} \tag{2.7}$$

together with $\bigwedge^p \mathcal{W} \perp \bigwedge^q \mathcal{W}$ for $p \neq q$. If X is a simple p-vector, $h(X|X)$ is its squared volume. Clifford multiplication respects h: if u is simple then

$$h(uX|uY) = h(u|u)h(X|Y) \qquad \forall X, Y \in \mathcal{C}\,.$$

The hat involution is an isometry with respect to h and the adjoint of left Clifford multiplication by X is multiplication by $\hat{X}$. The inner product on $\mathrm{End}\,\mathcal{S}$ given by

$$(\mathbf{A}, \mathbf{B}) \mapsto \tfrac{1}{8}\,\mathrm{Tr}\,\hat{\mathbf{A}}\mathbf{B}$$

makes the representation (2.2) an isometry:

$$h(X|Y) = \tfrac{1}{8}\,\mathrm{Tr}\,\widehat{\gamma(X)}\gamma(Y)\,. \tag{2.8}$$

Putting $X = \gamma^{-1}\mathbf{A}$ in this identity yields an inversion formula for the representation

$$(\gamma^{-1}\mathbf{A})^{I_p} = \frac{1}{8p!}\mathrm{Tr}\,(\gamma^{I_p} \circ \hat{\mathbf{A}}) \qquad (I_p \equiv i_1 \cdots i_p) \tag{2.9}$$

where X^{I_p} denotes the part of $X \in \mathcal{C}$ in $\bigwedge^p \mathcal{W}$. (The indices on the right-hand side have been raised with h^{-1}.) In particular, if

$$\mathbf{A} = \begin{pmatrix} 0 & U^{ab} \\ V_{ab} & 0 \end{pmatrix} \tag{2.10}$$

with U^{ab} and V_{ab} symmetric in ab, then $\gamma^{-1}\mathbf{A}$ lies in $\bigwedge^3 \mathcal{W}$ and

$$(\gamma^{-1}\mathbf{A})^{I} = \frac{1}{8 \times 3!}\left(\check{\Gamma}^{I}{}_{ab}U^{ab} + \Gamma^{I\,ab}V_{ab}\right) \qquad (I \equiv ijk)\,. \tag{2.11}$$

Conversely, if X lies in $\bigwedge^3 \mathcal{W}$ then $\gamma(X)$ is of the form (2.10) with U^{ab} and V_{ab} symmetric. It is probably worth noting how the above relations correspond to some of those in the appendix of Penrose and Rindler (1986) (where our $\mathbf{E}$ appears as $\varepsilon_{\rho\sigma}$). Their formula (B.29), together with (B.41), namely

$$\mathbf{E}^{-1} \otimes \mathbf{E} = \frac{1}{8}\mathbf{I} \otimes \mathbf{I} + \frac{1}{8} \sum_{p=1}^{6} \frac{1}{p!} \gamma_{i_1 \cdots i_p} \otimes \gamma^{i_1 \cdots i_p}$$

in which each side is considered in $\mathrm{End}\,(\mathcal{S} \otimes \mathcal{S})$, may be obtained by first rewriting the isometry identity (2.8) as

$$\mathrm{Tr}\,\hat{\mathbf{A}}\mathbf{B} = 8h\left(\gamma^{-1}\hat{\mathbf{A}} \middle| \gamma^{-1}\hat{\mathbf{B}}\right)$$

using (2.6) on the left-hand side, (2.7) and (2.9) on the right-hand side, and then extracting $\mathbf{A}$ and $\mathbf{B}$ from the result.

A nice basis of $\mathcal{W}$ for computations in $\mathcal{C}$ is obtained by choosing a basis $\{e_0, e_1, e_2, e_3\}$ of V normalized so that

$$\varepsilon(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = 1 \tag{2.12}$$

and defining

$$M_l := e_0 \wedge e_l \qquad \text{and} \qquad N_l := e_m \wedge e_n \tag{2.13}$$

where lmn varies over cyclic permutations of 123. Then $2h(M_l, N_m) = \delta_{lm}$. We associate with such a null basis an orthonormal basis

$$X_l := M_l - N_l \qquad \text{and} \qquad Y_l := M_l + N_l \,. \tag{2.14}$$

The trivector $X_1X_2X_3$ has inner product -1, spans a negative definite subspace of $\mathcal{W}$, and has a (numerical) matrix representative given by

$$\gamma(X_1X_2X_3) \sim \begin{pmatrix} 0 & \mathbf{I}_4 \\ -\mathbf{I}_4 & 0 \end{pmatrix}$$

with respect to the basis of $\mathcal{S}$ given by $\{e_m\}$ and its dual. The trivector $Y_1Y_2Y_3$ has inner product -1, spans a positive definite subspace, and has representative

$$\gamma(Y_1Y_2Y_3) \sim \begin{pmatrix} 0 & \mathbf{I}_4 \\ \mathbf{I}_4 & 0 \end{pmatrix} . \tag{2.15}$$

The identity matrix $\mathbf{I}_4$ of the upper right entry in (2.15) represents $\sum_{l=0}^{3} e_l \otimes e_l$, which is a type $(4,0)$ inverse metric whose associated dual operator has self-dual subspace the span of the Y_l; the lower left entry in (2.15) represents the metric. This will be explained in section 4.

We take the orientation of V to be given by any contravariant volume element whose contraction with ε is positive. We prescribe the orientation of $\mathcal{W}$ to be that of the volume element

$$8\, e_{01}\, e_{02}\, e_{03}\, e_{23}\, e_{31}\, e_{12} \in \bigwedge^6 \mathcal{W} \tag{2.16}$$

where $\{e_m\}$ is any oriented basis of V, $e_{mn} := e_m \wedge e_n$, and juxtaposition denotes Clifford multiplication (or exterior multiplication). If $\{e_m\}$ is normalized as in (2.12) and X_l and Y_l are defined as in (2.14), then (2.16) is the unit volume element

$$\eta = X_1 X_2 X_3 Y_1 Y_2 Y_3 \tag{2.17}$$

whose representative is

$$\gamma(\eta) = \begin{pmatrix} \mathbf{1}_V & 0 \\ 0 & -\mathbf{1}_{V^*} \end{pmatrix} . \tag{2.18}$$

There are two well known mappings providing (non-simply connected) double coverings of the identity component of the orthogonal group $\mathrm{O}(3,3) \cong \mathrm{O}(\mathcal{W}, h)$. To conclude this section we show that by way of the representation (2.2) these covering maps are essentially identical.

Proposition. The following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}_2 & \to & \mathrm{Spin}_o(\mathcal{W},h) & \xrightarrow{Ad_X} & \mathrm{SO}_o(\mathcal{W},h) & \to & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \to & \mathbf{Z}_2 & \to & \mathrm{SL}(V) & \xrightarrow{\Lambda^2} & \mathrm{SO}_o(\Lambda^2 V,h) & \to & 1\,. \end{array} \tag{2.19}$$

The subscripts o indicate that we are taking identity components of the groups. The surjective homomorphism in the top sequence is the adjoint representation, $Ad_X F = XFX^{-1}$; the surjective homomorphism in the bottom sequence is the wedging map, $(\wedge^2 A)(u \wedge v) = Au \wedge Av$; and the middle vertical map gives the $V \to V$ part of the representation γ, $X \mapsto \gamma(X)^a{}_b \in \mathrm{End}\,(V)$. The exactness of the rows in (2.19) is standard; and if $X \in \mathrm{Spin}_o(\mathcal{W},h)$ then $\gamma(X)^a{}_b$ lies in $\mathrm{SL}(V)$ (Harvey 1990, pp 203, 250). To show commutativity of the diagram we define

$$\mathbf{T}: \mathcal{W} \to \mathcal{S} \wedge \mathcal{S} \subset \mathrm{Hom}(\mathcal{S}^*, \mathcal{S}) \qquad \mathbf{T}(F) := \tfrac{1}{2}\gamma(F)\mathbf{E}^{-1} = \begin{pmatrix} F^{ab} & 0 \\ 0 & \hat{F}_{ab} \end{pmatrix} .$$

For $X \in \mathrm{Spin}_o(\mathcal{W},h)$ we have $X^{-1} = \hat{X}$ (Harvey, p 200) and (2.6) yields

$$\mathbf{T}(Ad_X F) = \gamma(X)\mathbf{T}(F)\gamma(X)^T \qquad \forall\, F \in \mathcal{W} .$$

The piece of this in $\Lambda^2 V \subset \Lambda^2 \mathcal{S}$ is

$$(Ad_X F)^{ab} = \gamma(X)^a{}_c \gamma(X)^b{}_d F^{cd} .$$

Thus the diagram (2.19) commutes. It should be noted that the middle vertical map is an isomorphism. The diagram, together with the obvious fact that -1 does not map to the identity, shows the kernel to be trivial; and the map is onto because the groups are connected and they have the same dimension. The fundamental groups of the middle groups are isomorphic to $\mathbf{Z}_2$.

3. The spinor geometry of the representation

We now discuss the relation between reduced spinors (i.e. vectors or covectors) in the representation (2.2) and totally null 3-planes in the space of bivectors. The intersections of certain such 3-planes are totally null 2-planes, which may be described by a vector–covector pair. To avoid overusing the word 'bivector', we shall use the appellations 'bivecteur' for an element of $\bigwedge^2 \mathcal{W}$ and 'trivecteur' for an element of $\bigwedge^3 \mathcal{W}$. Everything here holds in the complex case.

A subspace $\mathcal{N}$ of $\mathcal{W}$ is called *totally null* if every element of $\mathcal{N}$ is null with respect to h. The totally null subspace of $\mathcal{W}$ associated with a spinor $\xi \in \mathcal{S}$

$$\mathcal{N}(\xi) := \{F \in \mathcal{W} \mid \gamma(F)\xi = 0\}$$

is three-dimensional whenever ξ is a member of V or V^*. For $u \in V$, denote $\mathcal{N}(u)$ by $\alpha[u]$; we find

$$\alpha[u] = \{u \wedge v \mid v \in V\} = \{\tilde{\Gamma}^i{}_{ab} u^a v^b \mid v \in V\}$$

and call it the *α-triplane* of u. For $\phi \in V^*$, denote $\mathcal{N}(\phi)$ by $\beta[\phi]$; we find

$$\beta[\phi] = \Lambda^2 \ker \phi = \{\Gamma^{iab}\phi_a \psi_b \mid \psi \in V^*\}$$

and call it the *β-triplane* of ϕ. The triplanes $\alpha[u]$ and $\beta[\phi]$, as u and ϕ range over V and V^*, respectively, exhaust the three-dimensional totally null subspaces of $\mathcal{W}$. Two α-triplanes (or two β-triplanes) either coincide or they intersect in one dimension. An α-triplane and a β-triplane either intersect trivially or they intersect in two dimensions. Totally null triplanes and their intersections and sums have certain distinguished representatives in the exterior algebra, which we now describe.

We define the *α-trivecteur* of $u \in V$ to be

$$\alpha(u)^{ijk} := \frac{1}{8 \times 3!} \tilde{\Gamma}^{ijk}{}_{ab} u^a u^b \tag{3.1}$$

so that, according to (2.10) and (2.11)

$$\gamma(\alpha(u)) = \begin{pmatrix} 0 & u \otimes u \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

The trivecteur $\alpha(u)$ is simple and spans the α-triplane of u. If $\{u, k_1, k_2, k_3\}$ is a basis of V then

$$\alpha(u) = e^{-1}(u \wedge k_1)(u \wedge k_2)(u \wedge k_3) \tag{3.3}$$

where $e = \varepsilon(u \wedge k_1 \wedge k_2 \wedge k_3)$, which could be arranged to be 1, and where juxtaposition denotes Clifford multiplication (or exterior multiplication in $\bigwedge \mathcal{W}$). This may be seen by computing with the Cartan 3-form (2.4)

$$\Gamma(u \wedge k_1)^{ac}\, \tilde{\Gamma}(u \wedge k_2)_{cd}\, \Gamma(u \wedge k_3)^{db} = e\, u^a u^b .$$

One can obtain (3.1) in a somewhat more geometrical way by considering the map $u_\wedge : V \to \alpha[u]$ given by $v \mapsto u \wedge v$ and its wedge,

$$\bigwedge^2(u_\wedge) : \Lambda^2 V \to \bigwedge^2 \alpha[u]$$

which annihilates $\alpha(u)$ and thus determines an element of the tensor product of the annihilator of $\alpha(u)$ in $\mathcal{W}^*$ with $\bigwedge^2 \alpha[u]$. Using h^{-1} to 'raise the index' on the annihilator piece and skew-symmetrizing the result gives $\alpha(u)$ up to a constant factor.

We define the *β-trivecteur* of $\phi \in V^*$ to be

$$\beta(\phi)^{ijk} := \frac{1}{8 \times 3!} \Gamma^{ijk\,ab} \phi_a \phi_b \tag{3.4}$$

so that

$$\gamma(\beta(\phi)) = \begin{pmatrix} 0 & 0 \\ \phi \otimes \phi & 0 \end{pmatrix}. \tag{3.5}$$

If $\{k_1, k_2, k_3\}$ is a basis of $\ker \phi$ and $\phi(v) \neq 0$ then

$$\beta(\phi) = f\,(k_2 \wedge k_3)(k_3 \wedge k_1)(k_1 \wedge k_2) \tag{3.6}$$

where $f = \varepsilon(v \wedge k_1 \wedge k_2 \wedge k_3)^{-2} \phi(v)^2$.

Left multiplication by the volume element η leaves an α-trivecteur invariant and reverses the sign of a β-trivecteur, meaning these are self-dual and anti-self-dual, respectively, with respect to this duality operation in $\bigwedge^3 \mathcal{W}$. Inner products of α- and β-trivecteurs are easily computed using their representatives (3.2) and (3.5) in the isometry identity (2.8) together with the fact that the hat involution does not affect trivecteurs. For non-zero $u, v \in V$ and non-zero $\phi, \psi \in V^*$ we find

$$h(\alpha(u) \,|\, \alpha(v)) = h(\beta(\phi) \,|\, \beta(\psi)) = 0 \qquad \text{and} \qquad h(\alpha(u) \,|\, \beta(\phi)) = \tfrac{1}{8} \phi(u)^2 . \tag{3.7}$$

The vanishing of the inner product in (3.7) is equivalent to the intersection of $\alpha[u]$ and $\beta[\phi]$ being non-trivial and hence two-dimensional.

It is fairly easy to see that every totally null biplane in $\mathcal{W}$ arises as the intersection of an α-triplane and a β-triplane. Such an intersection can be represented algebraically. If $u \in V$ and $\phi \in V^*$ are non-zero and $\phi(u) = 0$, then

$$\gamma^{-1} \begin{pmatrix} u \otimes \phi & 0 \\ 0 & -\phi \otimes u \end{pmatrix} = \frac{1}{8} \Gamma^{ij}{}_a{}^b u^a \phi_b \tag{3.8}$$

and the right-hand side of (3.8) is a simple bivecteur spanning the totally null biplane $\alpha[u] \cap \beta[\phi] = \mathcal{N}\binom{u}{\phi}$. In fact, if

$$t \notin \operatorname{Span}\{u, v, w\} = \ker \phi$$

then

$$\gamma^{-1} \begin{pmatrix} u \otimes \phi & 0 \\ 0 & -\phi \otimes u \end{pmatrix} = \frac{\phi(t)}{e} (u \wedge v)(u \wedge w) \tag{3.9}$$

where $e = \varepsilon(t \wedge u \wedge v \wedge w)$. To verify (3.9) one calculates the representative of the right-hand side with the Cartan 2-form (2.3). To get (3.8), which lies in $\bigwedge^2 \mathcal{W}$ by (3.9), one uses the inversion formula (2.9) (cf Cartan (1981, section 124), who gives the conditions $\phi(u) = 0$ and the right-hand side of (3.8) being non-zero for $\alpha[u]$ and $\beta[\phi]$ to intersect non-trivially). The four-dimensional sum $\alpha[u] + \beta[\phi]$—also the subspace orthogonal to the intersection—is spanned by

$$\gamma^{-1} \begin{pmatrix} u \otimes \phi & 0 \\ 0 & \phi \otimes u \end{pmatrix} = 2 \frac{\phi(t)}{e^2} (u \wedge t)(u \wedge v)(u \wedge w)(v \wedge w)$$

where the juxtaposition is *exterior* multiplication in $\bigwedge \mathcal{W}$ (not the same as Clifford multiplication in this case).

4. Metrics and dual operators

In the two previous sections our basic data were simply a four-dimensional vector space V and a volume form ε, the latter involving only a choice of scale (this choice being immaterial to much of the spinor geometry). Into this arena one may consider introducing further structure on V and examining its effect in $\mathcal{W} = \Lambda^2 V$ (cf Urbantke 1989), for which the natural Clifford algebra representation may be a useful tool. Here we consider an inner product, or metric, on V; in this case induced structures in the Clifford algebra of $\mathcal{W}$ are enough to identify the metric up to sign.

Given a metric g on V, an associated dual operator $\mathcal{D}$ acting on bivectors is defined by

$$g_{(2)}(\mathcal{D}F, G) = \varepsilon(F \wedge G) \qquad \forall\, F, G \in \mathcal{W}$$

where $g_{(2)}$ is the induced inner product on $\Lambda^2 V$ defined in the manner of $h(\ |\ )$ in (2.7) and ε has been normalized so that $\varepsilon^{abcd}\varepsilon_{abcd} = \pm 24$, indices being raised with g^{-1}. This yields the formula

$$\mathcal{D}F = g^{-1}\check{F}g^{-1}$$

in which we consider g^{-1} and $\check{F}$ as appropriate homomorphisms; in index notation

$$(\mathcal{D}F)^{ab} = g^{ac}g^{bd}\check{F}_{cd} = \tfrac{1}{2}\varepsilon^{ab}{}_{cd}F^{cd}.$$

We also have

$$(\mathcal{D}F)\check{} = \pm gFg$$

the sign being that of $\mathcal{D}^2$. Such dual operators have exactly two eigenvalues, of opposite sign, each with a three-dimensional eigenspace. By factorizing $\gamma(\mathcal{D}F)$ in terms of matrices involving g, we shall see that the representatives under γ of the volume elements of the eigenspaces involve the metric in a simple way.

Suppose $\mathcal{D}^2 = \mathbf{1}$, which occurs for a metric g of definite or split signature type. Then $\mathcal{D}$ has eigenvalues ± 1 with corresponding eigenspaces $\mathcal{W}^+$ and $\mathcal{W}^-$.

Theorem. There exists a unique volume element Z of $\mathcal{W}^+$ such that

$$\gamma(Z) = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{4.1}$$

Proof. If we *define* Z to be the inverse under γ of the right-hand side of (4.1), then for any $F \in \mathcal{W}$

$$\gamma(\mathcal{D}F) = \gamma(Z)\gamma(F)\gamma(Z) \tag{4.2a}$$

and hence

$$\mathcal{D}F = ZFZ. \tag{4.2b}$$

To identify Z, we observe that, under the twisted adjoint representation $\widetilde{Ad}$, Z gives

$$\widetilde{Ad}_Z F = -ZFZ^{-1} = -\mathcal{D}F. \tag{4.3}$$

Since $\widetilde{Ad}_Z$ preserves $\mathcal{W}$ and $h(Z|Z) = 1$ by (2.8), $Z \in \mathrm{Pin}(\mathcal{W}, h)$ (Harvey 1990, p 203). Moreover, Z covers the reflection $-\mathcal{D} \in \mathrm{O}(\mathcal{W}, h)$ along the self-dual subspace $\mathcal{W}^+$. Now, there are exactly two elements in $\mathrm{Pin}(\mathcal{W}, h)$ that cover this reflection, namely the two unit volume elements of $\mathcal{W}^+$ (Harvey 1990, p 198). Thus Z is one of them. □

Some remarks: the identity (4.3) (valid for *any* volume element Z of $\mathcal{W}^+$) may be verified directly by using the fact that Z commutes with elements of $\mathcal{W}^+$ and anticommutes with elements of the anti-self-dual subspace $\mathcal{W}^-$. The dual operator itself, which reflects along $\mathcal{W}^-$, is covered in $\mathrm{Pin}(\mathcal{W}, h)$ by the two unit volume elements of $\mathcal{W}^-$, one of which is

$$Z' = \gamma^{-1}\begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix} \tag{4.4}$$

that is, $\mathcal{D} = \widetilde{Ad}_{Z'}$, which amounts to writing a factorization of $\gamma(\mathcal{D}F)$ slightly different from (4.2a). It follows from $h(Z|Z) = 1$ that $\mathcal{W}^+$ has signature type $(+++)$ or $(+--)$ with respect to h. For a metric of definite signature, $\mathcal{W}^+$ in fact has type $(+++)$ (Harnett 1989, section 3). If $\{e_m\}$ is an orthonormal basis for the metric, the Y_l of (2.14) form a basis of self-dual bivectors and $Z = Y_1Y_2Y_3$ satisfies (4.1). On the other hand, the X_l of (2.14) give $Z' = X_1X_2X_3$ satisfying (4.4). In the split signature case $\mathcal{W}^+$ has type $(+--)$. If $\{e_m\}$ is an oriented orthonormal basis for the metric then $Z = Y_1X_2X_3$ satisfies (4.1) and $Z' = X_1Y_2Y_3$ satisfies (4.4).

Now suppose $\mathcal{D}^2 = -1$, which occurs for a Lorenzian metric g. Then $\mathcal{D}$ has eigenvalues $\pm\mathrm{i}$ with corresponding eigenspaces $\mathcal{W}^+$ and $\mathcal{W}^-$ in the complexification $\mathcal{W}_{\mathrm{C}}$ of $\mathcal{W}$. Note that $\mathcal{D}$ is not a reflection but $\mathrm{i}\mathcal{D}$ is. To deal with this case we complexify the representation γ.

Theorem. There exists a unique volume element S of $\mathcal{W}^+$ such that

$$\gamma(S) = \begin{pmatrix} 0 & g^{-1} \\ \mathrm{i}g & 0 \end{pmatrix}. \tag{4.5}$$

Proof. Define S to be the inverse under γ of the right-hand side of (4.5). Then $S \in \bigwedge^3 \mathcal{W}_{\mathrm{C}}$ and, by considering $\gamma(\mathcal{D}F)$, we see that $\mathcal{D}F = SFS$ for any $F \in \mathcal{W}_{\mathrm{C}}$. Thus

$$\widetilde{Ad}_S F = -SFS^{-1} = \mathrm{i}\mathcal{D}F$$

which gives the reflection of F along the self-dual subspace $\mathcal{W}^+$. It follows that S is one of the two volume elements of $\mathcal{W}^+$ such that $S^2 = \mathrm{i}$. □

We chose S for convenience. Though $S \notin \mathrm{Pin}\mathcal{W}_{\mathrm{C}}$, we do have $\theta S \in \mathrm{Pin}\mathcal{W}_{\mathrm{C}}$, where $\theta = \mathrm{e}^{-\mathrm{i}\pi/4}$; for this,

$$\gamma(\theta S) = \begin{pmatrix} 0 & \tilde{g}^{-1} \\ \tilde{g} & 0 \end{pmatrix}$$

where $\tilde{g} = \theta^{-1}g$. A volume element such as S for $\mathcal{W}^+$ may be obtained as $S = S_1S_2S_3$, where $S_l = M_l - \mathrm{i}N_l$ and the M_l and N_l are defined as in (2.13) with $\{e_m\}$ being an oriented orthonormal basis for the metric. By considering $\gamma(\mathcal{D}F)$ directly for $F \in \mathcal{W}$, we find that

$$\mathcal{D}F = \eta ZFZ \tag{4.6}$$

where η is the unit volume element (2.17) of $\bigwedge \mathcal{W}$ and Z is again as in (4.1). The relation with S is

$$Z = \sqrt{2}\,\mathrm{Re}(\theta S) = \mathrm{Re}(S) + \mathrm{Im}(S)\,.$$

Note that $\bar{S} \in \mathcal{W}^-$ and that the volume element of $\mathcal{W}$ is $\eta = \mathrm{i}S\bar{S}$. With these facts one can verify directly that $\mathcal{D}F = \eta ZFZ$. We could have found the appropriate Z by direct computation; for example, with a basis $\{e_m\}$ and M_l, etc, as above, the g^{-1} part of $\gamma(Z)$ is

$$\begin{pmatrix} 0 & g^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_0 \otimes e_0 - e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3 \\ 0 & 0 \end{pmatrix}$$

which is the image under γ of

$$\begin{aligned} \alpha(e_0) - \alpha(e_1) - \alpha(e_2) - \alpha(e_3) &= M_1 M_2 M_3 - M_1 N_2 N_3 - N_1 M_2 N_3 - N_1 N_2 M_3 \\ &= \mathrm{Re}(S_1 S_2 S_3)\,. \end{aligned}$$

Let us be more explicit about the formula for a metric in terms of a self-dual volume element. In the case $\mathcal{D}^2 = 1$, from (4.1) and (2.4) we have

$$g_{ab} = \check{\Gamma}_{ijk\,ab} Z^{ijk} = \check{\Gamma}_{i\,ac} \Gamma_j{}^{cd} \check{\Gamma}_{k\,db} Z^{ijk}\,. \tag{4.7}$$

This can be written in terms of an arbitrary basis Q_1, Q_2, Q_3 of $\mathcal{W}^+$ as

$$g_{ab} = \frac{1}{3!} \frac{\epsilon^{\mathbf{ijk}}}{h(Q|Z)} \check{Q}_{\mathbf{i}ac} Q_{\mathbf{j}}{}^{cd} \check{Q}_{\mathbf{k}db} \tag{4.8}$$

this being a sum over permutations of 123 in the numerical indices $\mathbf{ijk}$, where $\epsilon^{\mathbf{ijk}}$ is alternating with $\epsilon^{123} = 1$ and where $Q = Q_1 Q_2 Q_3$. The quantity $h(Q|Z)$ is $\pm h(Q|Q)^{1/2}$ according to whether the orientation of Q is the same or opposite to that of Z; and the quantity $h(Q|Q)$, we recall, is the squared volume of Q:

$$h(Q|Q) = \det\left[h(Q_{\mathbf{i}}, Q_{\mathbf{j}})\right]$$

which is always positive, since $h(Z|Z) = 1$. If $\mathcal{D}^2 = -\mathbf{1}$ we start with (4.5) to obtain (4.8) again with S in place of Z. The metric's dependence on the initial choice of volume form $\varepsilon \in \Lambda^4 V^*$ makes it a tensor density,

$$g[\lambda\varepsilon] = \lambda^{1/2} g\,.$$

Formula (4.8) was given by Urbantke (1984).

There are other noteworthy ways of expressing the metric in terms of the dual operator. The dual operator has an obvious extension to an algebra automorphism of $\bigwedge \mathcal{W}$. (If $\mathcal{D}^2 = 1$ this extension is the unique Clifford algebra automorphism extending $\mathcal{D}$. If $\mathcal{D}^2 = -1$ this extension is the unique isomorphism of Clifford algebras extending $\mathcal{D}$ considered as an isometry from $(\mathcal{W}, h)$ into $(\mathcal{W}, -h)$—see e.g. Harvey 1990, p 182.) We find, most easily by using (4.2) and (4.6) and considering representatives in $\mathrm{End}\,\mathcal{S}$, that

$$\mathcal{D}\alpha(u) = \beta(gu) \tag{4.9a}$$

and

$$\mathcal{D}\beta(\phi) = \pm\alpha(g^{-1}\phi) \tag{4.9b}$$

the sign being that of $\mathcal{D}^2$. Letting $\tilde{\alpha}$ and $\tilde{\beta}$ be the *linear* maps on the symmetric products obtained naturally from α and β, we obtain a commutative diagram of linear maps

$$\begin{array}{ccccc} & V\odot V & \xrightarrow{\tilde{\alpha}} & \bigwedge^3 W & \\ g\otimes g & \downarrow & & \downarrow & \mathcal{D} \\ & V^*\odot V^* & \xrightarrow{\tilde{\beta}} & \bigwedge^3 W & \end{array}$$

This exhibits the intertwining of metric and dual operator mentioned in the introduction. The results (4.9) taken with (3.7) lead also to the formulae

$$g(u,v)^2 = 8h(\alpha(u)\,|\,\mathcal{D}\alpha(v)) \tag{4.10a}$$

$$= \pm g_{(2)}(\alpha(u)\,|\,\alpha(v)) \tag{4.10b}$$

where $g_{(2)}(\ |\ )$ is the extension to $\bigwedge W$ of the inner product $g_{(2)}$, this extension defined in the manner of the extension $h(\ |\ )$ in (2.7), and the sign is again that of $\mathcal{D}^2$. (In the definite and split metric signature cases, all α-triplanes are of type $(+++)$ and $(+--)$, respectively, with respect to $g_{(2)}$; in the Lorentzian metric case, all α-triplanes are of type $(---)$ or $(++-)$ with respect to $g_{(2)}$; so, at least when $u = v$, one can see that the right-hand side of (4.10b) is non-negative for a real metric g.) We note from (4.10*a*) and the remark following (3.7) that $g(u,v) = 0$ if and only if the intersection of $\alpha(u)$ and $\mathcal{D}\alpha(v)$ is non-trivial; and from (4.10*b*) that $|g(u,u)|$ is the $g_{(2)}$-volume of $\alpha(u)$.

It is not difficult to see the action of a dual operator in the projective space of V. The four points determined by an orthonormal basis span a tetrahedron whose edges represent bivectors; the three edges passing through $[e_l]$ represent the α-triplane of e_l and the face opposite $[e_l]$ represents the β-triplane of $g(e_l)$. The dual operator maps each edge to the edge not meeting it. A vector u is null if and only if the dual operator maps some projective line through $[u]$ into another such line (and hence, because $\alpha[u]\cap\beta[gu]$ is then two-dimensional, the dual operator necessarily maps an entire projective plane through $[u]$ into itself.)

5. Applications

We briefly outline some applications. For each metric signature type and for the complex case, there is a correspondence between the set of metrics and a certain set of dual operators. These correspondences are made precise in Harnett (1991), but the proofs there involve special choices of bases of bivectors. With the perspective gained here, those correspondences can be derived more elegantly.

A well-known self-dual formalism for four-dimensional Lorentzian geometry has been used in connection with classification of Weyl curvature tensors and the study of null congruences (e.g. Israel 1979). Its effectiveness depends on dealing only with self-dual bivectors, but the natural representation of the full bivector Clifford algebra can add a helpful perspective and it provides scope for working with only real quantities. Passing to the self-dual formalism from the full Clifford algebra involves a volume element for the self-dual subspace such as S in (4.5). (Many identities in Israel's monograph can be simply understood by restricting identities in the Clifford algebra representation to the self-dual sector.) In fashioning a formulation of real general relativity without a metric (Capovilla *et al* 1991b) the real Clifford algebra is relevant. (The result of the appendix of that paper may be recognized as a generalization to $\mathrm{Pin}(\mathcal{W}, h)$ of the proposition of section 2 herein.) One may look at four-dimensional geometry from the perspective of the bivector bundle with a dual operator (equivalently, with a distinguished three-dimensional sub-bundle non-degenerate with respect to the canonical conformal structure) rather than the tangent bundle with a metric. Though it may be cumbersome, it is reasonable to consider connections on the principal SO(3, 3)-bundle of frames for the bivector bundle. Corresponding to the canonical 1-form θ^a of the SL(4, $\mathbf{R}$)-bundle of tangent frames one has a canonical form, essentially $\theta^a \wedge \theta^b$, on the SO(3, 3)-bundle; corresponding to torsion 2-form Θ^a one finds what is essentially $\Theta^{[a} \wedge \theta^{b]}$ on the SO(3, 3)-bundle.

In the recent self-dual two-form formulation of general relativity (Capovilla *et al* 1991a and Frauendiener and Mason 1990), one starts with a triple of self-dual 2-forms represented by a symmetric two-component-indexed 2-form Σ^{AB} satisfying a certain condition. The condition turns out to be exactly that $F \mapsto \Sigma_A{}^B(F)$ is a Clifford map for the self-dual subspace. Assuming another such Clifford map for the anti-self-dual subspace, it is somewhat curious that the two resulting spin spaces can be identified with the two half-spin spaces for the tangent space. We can see how this arises as follows. From the two Clifford maps one can build a representation for the full Clifford algebra—after the manner described in Penrose and Rindler (1986, p 458)—which must be equivalent to the natural representation on $V \oplus V^*$. One of the block matrix components of an intertwining map that effects the equivalence gives an isomorphism of V with the tensor product of the two spin spaces. That this isomorphism yields a Clifford map for V is a consequence of an identity (to be expected) involving the intertwining map, the inner product structures on the two spin spaces and the inner product $\mathbf{E}$ on the space $V \oplus V^*$ of the natural representation.

In the complex case, the relation established between dual operators and metrics yields an account of some projective geometry that (besides its intrinsic interest) pertains to the twistor description of momentum and angular momentum. Let $\mathbf{T}$ be four-dimensional complex vector space equipped with a volume form ε. The set of simple elements in $\Lambda^2\mathbf{T}$ determines a 4-quadric in $\mathbf{P}(\Lambda^2\mathbf{T})$, which we identify with the Grassmannian $\mathbf{M}$ of 2-planes in $\mathbf{T}$ via the Plücker correspondence, $[w \wedge z] \leftrightarrow \mathrm{sp}\{w, z\}$. Now introduce a non-degenerate symmetric bilinear form A on $\mathbf{T}$—a 'metric'; but also, in the setting of twistor theory (Penrose and Rindler 1986), a 'kinematic twistor' representing an elementary particle. The form A determines the set $\mathcal{C}$ of $X \in \mathbf{M}$ such that $A(w, z) = 0$ for all $w, z \in X$. $\mathcal{C}$ is known to be the disjoint union of two conics ($\mathbf{C}P^1$s) that parametrize the two families of projective lines that generate the 2-quadric $\mathcal{Q}$ in $\mathbf{PT}$ defined by $A(z, z) = 0$ (Hughston and Hurd 1983, p 309). One of the conics may be viewed as the complex mass-centre world line of the elementary particle represented by A. One can show, using (4.9), that $\mathcal{C}$ is the trace in $\mathbf{M}$ of the fixed point set of the projectivized dual operator $\mathcal{D}$ (with $\mathcal{D}^2 = 1$) associated to A. (To see a connection between the two conics making up $\mathcal{C}$, it is worth noting that each $z \in \mathcal{Q}$ determines a $\mathcal{D}$-invariant projective line $\alpha[z] \cap \beta[Az]$ in $\mathbf{P}(\Lambda^2\mathbf{T})$ meeting each conic once, these meetings occurring at the points X and Y for which the projective lines $\mathbf{P}X$ and $\mathbf{P}Y$ in $\mathbf{PT}$ meet at z.)

Thus a non-degenerate kinematic twistor A determines an involution $\mathcal{D}$ of $\mathbf{M}$ that fixes the world line of the represented particle. We can ask to see explicitly how $\mathcal{D}$ acts on points in an affine Minkowski space in $\mathbf{M}$. The result for ordinary non-singular kinematic twistors (e.g. equation (6.3.11) of Penrose and Rindler 1986) is that $\mathcal{D}$ reflects along the hyperplanes perpendicular to the 4-momentum and across the complex mass-centre world line. For a particle with no intrinsic spin, this world line is the real world line of the particle. For a particle with intrinsic spin this 'world line' determined by A is displaced from the real world line by i times a vector proportional to the Pauli–Lubanski spin vector (Penrose and Rindler 1986, pp 419–420), but the description of $\mathcal{D}$ is the same. Certain non-degenerate forms A represent uniformly accelerated particles (Harnett 1989); for such A the mass-centre world line consists of the two branches of a timelike hyperbola and there is a distinguished point O in real affine Minkowski space between the branches. In this case $\mathcal{D}$ is the inversion $x \mapsto x/(a^2x^2)$ in O (where a is the acceleration parameter and x is the displacement from O) followed by the reflection through O along the osculating plane of the hyperbola.

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